On the expected diameter of an L_2-bounded martingale

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- Ratio of exp diameter of L_2-bdd MG and the std of its last term cannot exceed $\sqrt{3}$.
- Exhibit a one-parameter family of stopping times on sBM for which the $\sqrt{3}$ upper bound is attained.
- Optimal when payoff for stopping at t is diameter $D(t)$ minus the accumulated cost ct
- *Maximal drawdown* - its expectation is bounded by $\sqrt{2}$ times the std of the last term.
- Dubins & Schwarz bounds 1 and $\sqrt{2}$ for ratios of exp maximum and maximal absolute value of the MG and the std of its last term.

Dubins & Schwarz: ratio between $E[M]$ of a mean-zero L_2-bdd MG (u.i., well defined terminal element) and the std of its last term is bounded above by 1. Attained by MG $\{X(t) = B(t \wedge \tau) : t \geq 0\}$

$$\tau = \tau_d = \inf\{t \geq 0 : M(t) - B(t) \geq d\} \ ,$$

the first time B displays a drawdown of size d

Purpose: demonstrate that B stopped at time

$$\mathcal{T}_d = \inf\{t \geq 0 : (M(t) - B(t) \wedge (B(t) - m(t)) \geq d\}$$

attains the least upper bound $\sqrt{3}$ on the ratio of the expected diameter $(D = M - m)$ to the std of the last term of any L_2-bdd MG.

$\mathcal{T}$ implemented in two stages:

Wait until diameter of size $2d$ is obtained; B is at maximum or minimum

if up, continue until a drawdown of size d is displayed;

if down, until a rise of size d is displayed.

Stage 1. RW S with equally likely $\pm\epsilon$ increments. From time to achieve diameter $h-\epsilon$ to time to achieve h - first exit from interval of length $h+\epsilon$ starting ϵ from end-point.
Expected incremental time $h\epsilon$ adds up to $\frac{h(h+\epsilon)}{2}$ as expected time to achieve diameter h.
Hence, $\frac{h^2}{2}$ for BM, $2d^2$ for $h = 2d$.

Stage 2.

Skorokhod embedding of $\text{Exp}(\frac{1}{d}) - d$ by Azéma-Yor stopping time:

$$0 = E[\text{MAX} - d] \rightarrow E[\text{MAX}] = d$$

$$\text{MAX} \sim \text{Exp}(\frac{1}{d}) \rightarrow E[\text{TIME}] = \text{Var}[\text{MAX - d}] = d^2$$

Achieved expected diameter $2d+d = 3d$ with expected time $2d^2 + d^2 = 3d^2$ and $\frac{3d}{\sqrt{3d^2}} = \sqrt{3}$.

Dubins & Schwarz - inequality for expected supremum S of a nonnegative L_2-bdd subMG (Gilat 1977: same as absolute value of a MG): least upper bound on the ratio of $E[S]$ to L_2-norm (square-root of the second moment) of last term is $\sqrt{2}$.

Bound attained by $|B|$ stopped at time

$$T = T_d = \inf\{t \geq 0 : S(t) - |B(t)| \geq d\}$$

where $S(t)$ is the supremum of $|B|$ on $[0, t]$.

One-sided diameters, *maximal drawdown* D^+ and *maximal rise* D^- (with $D = D^+ \vee D^-$)

$$D^+ = \sup_{t\geq 0}\{X(t) - \inf_{s>t} X(s)\} = \sup_{t\geq 0}\{M_X(t) - X(t)\}$$

Supremum over all L_2-bdd MGs X of ratio of $E[D^+]$ to the std of the last term of X is $\sqrt{2}$. Attained by MG $\{B(t) : t \leq \mathcal{T}^+\}$,

$$\begin{aligned} \mathcal{T}^+ = \mathcal{T}_d{}^+ &= \inf\{t \geq 0 : \sup_{s\leq t} A(s) - A(t) \geq d\} \\ &= \inf\{t > \tau_d : B(t) - \inf_{\tau_d < s \leq t} B(s) \geq d\} \end{aligned}$$

is earliest time *drop process* $A(t) = M(t) - B(t)$ attains a drawdown of size d.

Earliest time B attains rise d after drop d.

$\{B^2(t) - t : t \geq 0\}$ is mean-zero MG, Var $B(t) = t$

The c-problem (Dubins & Schwarz): maximizing desired ratios is related to finding optimal stopping time on B for payoff function $R(t) - ct$, $c > 0$ being the cost per unit time of sampling.

$R(t)$ can be $M(t)$, $m(t)$, $S(t)$, $D(t) = M(t) - m(t)$ and its two one-sided versions.

c-problem formulated as a continuous time dynamic programming (or gambling) problem, for which a toolkit is readily available.

Comment on relevance of Brownian Motion.

Variety of MG inequalities have as extremes segments of Brownian Motion up to stopping times.

Brownian Motion - universal embedding MG:

Skorokhod 1965: embedding of Z with $E[Z] = 0$ and $E[Z^2] < \infty$ in Brownian Motion by a stopping time T with $B(T) \sim Z$ and $E[T] = E[Z^2]$

Monroe 1972: for right-cont, mean-zero, L_2-bdd MG X, there exists increasing family $\{T_t : t \geq 0\}$ of *minimal* stopping times such that the process $\{B(T_t)\}$ has the same distribution as X

By L_2-bddness, the limiting stopping time $T = \lim_{t\to\infty} T_t$ is minimal and $B(T)$ has the same distribution as the last term of X

Maximum diameter of Brownian path up to time T dominates respective quantity in any embedded process.

Consequently, enough to establish inequalities for Brownian Motion stopped at minimal stopping times.

Theorem

*(*i*)* $\mathcal{T} = \mathcal{T}_{\frac{1}{2c}}$ optimal for c-problem with $R(t) = D(t)$.

*(*ii*)* $E[\mathcal{T}] = \frac{3}{4c^2}$

*(*iii*)* Optimal expected payoff $E[D(\mathcal{T}) - c\mathcal{T}] = \frac{3}{4c}$

Corollary

Expected diameter of L_2-bdd MG cannot exceed $\sqrt{3}$ times the std of its last term.

Upper bound $\sqrt{3}$ attained by segment of sBM between zero and any of the stopping times $\mathcal{T}_d$.

Similar theorems and corollaries for D^+ via $\mathcal{T}^+$.

Real-valued continuous function $q = q_{c,d}$ on the domain $\{(\delta, \gamma, t) : 0 \leq \gamma \leq \frac{\delta}{2} < \infty, t \geq 0\}$ in $\mathcal{R}^3$

$$q(\delta, \gamma, t) = \delta - ct +$$

$$\begin{cases} 0 & \gamma \geq d \\ 3d - \delta - c\{\gamma(\delta - \gamma) + 3d^2 - \frac{\delta^2}{2}\} & \delta < 2d \\ (d - \gamma)[1 - c(d + \gamma)] & \delta \geq 2d, \gamma < d \end{cases}$$

$D(t) = M(t) - m(t)$ diameter of B by time t
$G(t) = (M(t) - B(t)) \wedge (B(t) - m(t))$ is *gap*, minimal distance of current position from the extremal points visited so far.

Set payoff $\Pi(t) = D(t) - ct$ and consider process $Q(t) = Q_{c,d}(t) = q_{c,d}(D(t), G(t), t)$

Q is cond exp payoff under $\tau_{c,d,t}$ for c-problem given $\{B(s) : s \leq t\}$ with diameter $D(t)$ and gap $G(t)$:

If $G(t) \geq d$, $\tau_{c,d,t} = t$; otherwise,

First time after t at which gap G is at least d.

I.e., $\tau_{c,d,t}$ extends $\mathcal{T}_d$ to general initial conditions.

$\mathcal{T}_d$, with $d = \frac{1}{2c}$, optimal for c-problem because:

- Q majorizes the payoff Π
- $Q(0)$ is the expected payoff when using $\mathcal{T}_d$
- Q is a superMG

Thus, for every integrable stopping time τ,

$$E[\Pi(\tau)] \leq E[Q(\tau)] \leq Q(0) = E[\Pi(\mathcal{T}_d)]$$

Q superMG same as q *excessive* in Gambling terms of Dubins & Savage 1965, 1976

Proof consists of representing Q as piece-wise MG or superMG between properly defined stopping times.

$$Q = \frac{3}{4c} + \frac{c}{2}\{[(M(t) - B(t))^2 - t] + [(B(t) - m(t))^2 - t]\}$$

$$Q = \frac{1}{4c} + B(t^*) - m(\tau_1) + c[(M(t^*) - B(t^*))^2 - t^*]$$

$$t^* = \max(\tau_1, \min(\tau_2, t))$$

$$Q = M*(\tau) - m(\tau) - c\tau + c[\max(W(t^*), 0)^2 - (t^* - \tau)]$$

$$W(\cdot) = B(\cdot) - B(\tau)$$

Some stochastic inequalities of similar nature

$$P(MAX \geq k\sigma) \leq \frac{1}{k^2+1}$$

$$P(MAXABS \geq k\sigma) \leq \frac{1}{k^2}$$

(1 if $k \leq 1$, simply Chebyshev-Kolmogorov)

$$P(MAXDIAM \geq k\sigma) \leq \frac{1}{k^2-1}$$

(1 if $k \leq \sqrt{2}$)

Open problem: Are dominated locally fair processes martingales?

There ought to be a theorem to the effect that a dominated process that leaves small enough intervals (sub) fairly is a (super) MG, like our Q process.

Counter-example of Jim Pitman: uniform integrability (rather than domination) is not enough.

Open problem: the spider process

Larry Shepp reminded us that the $\sqrt{3}$-inequality is a special case of the *spider* problem raised by Dubins.

BM is abs BM each of whose excursions has random sign. The spider process with $n \geq 3$ rays is the extension from BM ($n = 2$) to an n-valued sign.

Maximal distance from origin is maximal absolute value of BM, independently of n.

Sum of distances from origin along the rays, reduces for $n = 2$ to diameter of BM.

Maximization of expected value of this sum for $n \geq 3$ seems harder to handle and requires new ideas.

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