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ON THE EXPECTED DIAMETER OF AN L_2-BOUNDED MARTINGALE

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Dedicated to the memory of Gideon Schwarz (1933–2007)

It is shown that the ratio between the expected diameter of an L_2-bounded martingale and the standard deviation of its last term cannot exceed $\sqrt{3}$. Moreover, a one-parameter family of stopping times on standard Brownian motion is exhibited, for which the $\sqrt{3}$ upper bound is attained. These stopping times, one for each cost-rate c, are optimal when the payoff for stopping at time t is the diameter $D(t)$ obtained up to time t minus the hitherto accumulated cost ct. A quantity related to diameter, *maximal drawdown* (or *rise*), is introduced and its expectation is shown to be bounded by $\sqrt{2}$ times the standard deviation of the last term of the martingale. These results complement the Dubins and Schwarz respective bounds 1 and $\sqrt{2}$ for the ratios between the expected maximum and maximal absolute value of the martingale and the standard deviation of its last term. Dynamic programming (gambling theory) methods are used for the proof of optimality.

1. Introduction. Lester Dubins and Gideon Schwarz [3] prove that the ratio between the expectation of the maximum M of a mean-zero L_2-bounded martingale (thus, uniformly integrable, with a well-defined terminal element) and the standard deviation (L_2-norm) of its last term is bounded above by 1. They go on to show that this bound is attained by the martingale $\{B(t): t \le \tau\}$, where the process $B = \{B(t): t \ge 0\}$ is standard Brownian motion and $\tau = \tau_d$, given by

$$\tau_d = \inf\{t \ge 0: M(t) - B(t) \ge d\}, \tag{1.1}$$

is the first time B displays a *drawdown* of size d, that is, B drops d units below the highest position it has visited so far; here $M(t)$ is the maximum of B on $[0, t]$ while d is any positive constant.

Clearly, the dual stopping time τ' for minimizing the expected minimum m relative to the standard deviation of the last term would be $\tau' = \tau'_d = \inf\{t \ge 0: B(t) - m(t) \ge d\}$, $m(t)$ being the minimum of B on $[0, t]$. τ' is the first time B displays a *rise* of size d.

The main purpose of the present article is to demonstrate that B stopped at time

$$\mathcal{T} = \mathcal{T}_d = \inf\{t \ge 0: (M(t) - B(t)) \wedge (B(t) - m(t)) \ge d\} \tag{1.2}$$

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attains the least upper bound, whose value will be shown to be $\sqrt{3}$, on the ratio between the expected diameter ($D = M - m$) and the standard deviation of the last term of any L_2-bounded martingale.

It is useful to point out that the stopping time $\mathcal{T}$ can be implemented in two stages: first, wait until for the first time a diameter of size $2d$ is obtained. At this moment B must be either at its hitherto maximum (up) or minimum (down). If it is up, continue until from that time on a drawdown of size d is displayed; similarly, if it is down, continue until a rise of size d is displayed. It is easy to check that this two-stage procedure terminates exactly at time $\mathcal{T}$.

Dubins and Schwarz [3] consider also the analogous inequality for the expected supremum S of a nonnegative L_2-bounded submartingale (which by Gilat [5] is the same as the absolute value of a martingale), proving that the least upper bound on the ratio between $E[S]$ and the L_2-norm (square-root of the second moment) of the last term is $\sqrt{2}$. Moreover, as they show, this bound is attained by the absolute value $|B|$ of B, stopped at time

$$T = T_d = \inf\{t \geq 0 : S(t) - |B(t)| \geq d\}, \tag{1.3}$$

where $S(t)$ is the supremum of $|B|$ on $[0, t]$ and as before, d is any positive constant.

Rephrasing S and D in terms of M and m by setting $S = M \vee |m|$ and $D = M - m = M + |m|$, it is seen that the stopping time τ is optimal for M, its dual τ' is optimal for $|m|$, T is optimal for the maximum $M \vee |m|$ of M and $|m|$ and $\mathcal{T}$ for their sum $M + |m|$. *Optimal* here means maximizing the pertinent ratio of expectation to standard deviation. The respective least upper bounds for these ratios are $1, 1, \sqrt{2}$ and $\sqrt{3}$, the last being the main contribution of the present paper.

Related to the diameter D of a process X [with $X(0) = 0$] are its one-sided versions, the *maximal drawdown* D^+ and the *maximal rise* D^- (with $D = D^+ \vee D^-$) defined in terms of $M_X(t) = \sup_{s \leq t} X(s)$ and $m_X(t) = \inf_{s \leq t} X(s)$, as follows:

$$\begin{aligned} D^+ &= \sup_{t \geq 0} \Big\{ X(t) - \inf_{s > t} X(s) \Big\} = \sup_{t \geq 0} \{M_X(t) - X(t)\}, \\ D^- &= \sup_{t \geq 0} \Big\{ \sup_{s > t} X(s) - X(t) \Big\} = \sup_{t \geq 0} \{X(t) - m_X(t)\}. \end{aligned} \tag{1.4}$$

It will be shown (Theorem 2 and Corollary 2) that the supremum, over all L_2-bounded martingales X, of the ratio between $E[D^+]$ (similarly for $E[D^-]$) and the standard deviation of the last term of X is $\sqrt{2}$. Furthermore, this bound is attained by the martingale $\{B(t) : t \leq \mathcal{T}^+\}$, where

$$\begin{aligned} \mathcal{T}^+ = \mathcal{T}_d{}^+ &= \inf\Big\{ t \geq 0 : \sup_{s \leq t} A(s) - A(t) \geq d \Big\} \\ &= \inf\Big\{ t > \tau_d : B(t) - \inf_{\tau_d < s \leq t} B(s) \geq d \Big\} \end{aligned} \tag{1.5}$$

is the earliest time the *drop process* $A(t) = M(t) - B(t)$ attains a drawdown of size d. Equivalently, $\mathcal{T}^+$ is the earliest time B attains a rise of size d after having had a drawdown of size d.

Recalling that the variance of $B(t)$ is t [in fact, $\{B^2(t) - t : t \geq 0\}$ is a mean-zero martingale], it can be seen (as also observed in Dubins and Schwarz [3] for maximizing the expected maximum) that the problem of maximizing the desired ratios is closely related to that of finding an optimal stopping time on B for the payoff function $R(t) - ct$, $c > 0$, being the cost per unit time of sampling. For brevity, refer to this as the *c-problem*. Here the *reward function* $R(t)$ can be any of the quantities $M(t)$, $m(t)$, $S(t)$, $D(t) = M(t) - m(t)$ and its two one-sided versions. In fact, it is not hard to obtain the solution to the ratio-maximization problem from the solution to the corresponding c-problem and vice versa. We choose to focus on the latter because it can be conveniently formulated as a continuous time dynamic programming (or gambling) problem, for which a toolkit is readily available.

REMARK. Dubins and Schwarz [3] solve the ratio-maximization problem for M directly and then infer the solution to the corresponding c-problem. For S, however, they go only in the opposite direction. A direct solution to the ratio-maximization for S can be found in Gilat [6]. We do not know how to solve the ratio-maximization problem for D other than by first solving the corresponding c-problem.

Recalling the definition (1.2) of the stopping time $\mathcal{T}$, our main result now follows:

THEOREM 1. (i) *For each $c > 0$, $\mathcal{T} = \mathcal{T}_{1/(2c)}$ is optimal for the c-problem with reward function $R(t) = D(t)$. It is the unique optimal stopping time within the family $\{\mathcal{T}_d : d > 0\}$.*

(ii) $E[\mathcal{T}] = \frac{3}{4c^2}$.

(iii) *The optimal expected payoff is $E[D(\mathcal{T}) - c\mathcal{T}] = \frac{3}{4c}$.*

COROLLARY 1. *The expected diameter of any L_2-bounded martingale cannot exceed $\sqrt{3}$ times the standard deviation of its last term. The upper bound $\sqrt{3}$ is attained by the segment of Brownian motion between zero and any of the stopping times $\mathcal{T}_d$.*

To state the next theorem, let $D^+(t)$ be the D^+ variable [see (1.4)] defined on the martingale $X(\cdot) = B(\cdot \wedge t)$, and recall the definition (1.5) of the stopping time $\mathcal{T}^+$.

THEOREM 2. (i) *For each $c > 0$, $\mathcal{T}^+ = \mathcal{T}^+_{1/(2c)}$ is optimal for the c-problem with reward function $R(t) = D^+(t)$. It is the unique optimal stopping time within the family $\{\mathcal{T}_d{}^+ : d > 0\}$.*

(ii) $E[\mathcal{T}^+] = \frac{2}{c^2}$.

(iii) *The optimal expected payoff is* $E[D^+(\mathcal{T}^+) - c\mathcal{T}^+] = \frac{1}{2c}$.

COROLLARY 2. *The expected maximal drawdown of any L_2-bounded martingale cannot exceed $\sqrt{2}$ times the standard deviation of its last term. The upper bound $\sqrt{2}$ is attained by the segment of Brownian motion between zero and any of the stopping times $\mathcal{T}_d{}^+$.*

Comment on the special relevance of Brownian motion. It should not be surprising that in a variety of martingale inequalities (those considered here included), the extremal martingales, namely those for which equality is attained, are segments of Brownian motion determined by a suitable stopping time. Moreover, in order to establish an inequality for the class of L_2-bounded martingales, it typically suffices to consider the subclass of these processes of the form $\{B(t) : t \le T\}$, where T is a stopping time with $E[T] < \infty$. This is so simply because Brownian motion is a universal martingale in the following very specific sense. Recall the Skorokhod [12] embedding of a random variable Z with $E[Z] = 0$ and $E[Z^2] < \infty$ in Brownian motion by a stopping time T, such that $B(T) \sim Z$ and $E[T] = E[Z^2]$; following I. Monroe [10] call such a stopping time *minimal* (for Z). Monroe ([10], Theorem 11) extends Skorokhod's result as follows: given a right-continuous, mean-zero, L_2-bounded martingale X, there exists an increasing family $\{T_t : t \ge 0\}$ of minimal stopping times such that the embedded process $\{B(T_t)\}$ has the same distribution as X. By L_2-boundedness it follows that the limiting stopping time $T = \lim_{t\to\infty} T_t$ is minimal and that $B(T)$ has the same distribution as the last term of X. Note also that a process in discrete time can always be extended to continuous time and made right-continuous by setting it constant between consecutive integer time points. Clearly, the maximum or the diameter of the entire Brownian path up to time T dominates the respective quantities in any embedded process. Consequently, it is enough to establish our inequalities for Brownian motion stopped at minimal stopping times.

2. Excessivity and supermartingales—Proofs.

2.1. *Proof of Theorem 2 and its corollary.* Recalling the definition of $D^+(t)$ preceding the statement of Theorem 2 and the definition of the drop process A following (1.5), observe that $D^+(t) = \sup_{s\le t} A(s)$. Since A is distributed like the absolute value of a Brownian motion (see Karatzas and Shreve [8], page 97, who attribute this result to Paul Lévy), the c-problems for maximal drawdown and maximal absolute value are equivalent. Consequently, Theorem 2 and its corollary follow from the $\sqrt{2}$-inequality of Dubins and Schwarz [3] (quoted in the Introduction) regarding the absolute value of a martingale.

2.2. *Proof of Theorem* 1. Define a real-valued function $q = q_{c,d}$ on the domain $\{(\delta, \gamma, t): 0 \le \gamma \le \frac{\delta}{2} < \infty, t \ge 0\}$ in $\mathcal{R}^3$ by

$$q(\delta, \gamma, t) = \delta - ct + \begin{cases} 0, & \gamma \ge d, \\ 3d - \delta - c\Big\{\gamma(\delta - \gamma) + 3d^2 - \dfrac{\delta^2}{2}\Big\}, & \delta < 2d, \\ (d - \gamma)[1 - c(d + \gamma)], & \delta \ge 2d, \gamma < d. \end{cases} \tag{2.1}$$

Note that q is a continuous function.

Let $D(t) = M(t) - m(t)$ be the diameter attained by B by time t and let $G(t) = (M(t) - B(t)) \wedge (B(t) - m(t))$ be the *gap*, or minimal distance of the current position from the extremal points visited so far. Consider the process $Q(t) = Q_{c,d}(t) = q_{c,d}(D(t), G(t), t)$ and set $\Pi(t) = D(t) - ct$, the payoff function.

With the help of Lemmas 1 and 2 and Corollary 3 below, Q can be identified as the conditional expected payoff for the c-problem [with reward $R(t) = D(t)$] given a partial history $\{B(s): s \le t\}$ with current diameter $D(t)$ and gap $G(t)$, when the following stopping time $\tau_{c,d,t}$ is used: If $G(t) \ge d$, $\tau_{c,d,t} = t$; otherwise, $\tau_{c,d,t}$ is the first time after t at which the gap G is at least d. In other words, $\tau_{c,d,t}$ extends $\mathcal{T}_d$ [see (1.2)] to general initial conditions. That $\mathcal{T}_d$, with $d = \frac{1}{2c}$, is optimal for the c-problem will follow from properties of the Q process to be established in Proposition 1: Q majorizes the payoff Π, $Q(0)$ is the expected payoff when using $\mathcal{T}_d$ and Q is a supermartingale. Thus, for every integrable stopping time τ, $E[\Pi(\tau)] \le E[Q(\tau)] \le Q(0) = E[\Pi(\mathcal{T}_d)]$. Q being a supermartingale is the same as q being *excessive* in the gambling theoretic terminology of Dubins and Savage [2], a notion closely related to *no-arbitrage pricing* in finance (see, e.g., page 92 in Dana and Jeanblanc [1]).

The following two lemmas summarize known results, most of which are used in the sequel.

LEMMA 1. *Recall* (1.1) *and let* $\varepsilon_{a,b,x}$, *with* $a \le x \le b$, *be the first exit time from the interval* (a, b) *by Brownian motion starting at* x.

(i) (Common knowledge, see, e.g., [4], page 71). $E[\varepsilon_{a,b,x}] = (x - a)(b - x)$ *and* $P(B(\varepsilon_{a,b,x}) = a) = \frac{b-x}{b-a}$. *Similarly for simple random walk when* a, b *and* x *are integers*.

(ii) (Dubins and Schwarz [3]). $M(\tau_d) = B(\tau_d) + d$ *is exponentially distributed with mean* d. *Hence*, $E[M(\tau_d)] = d$ *and* $E[\tau_d] = \mathrm{Var}[B(\tau_d)] = d^2$.

LEMMA 2. *Let* δ_h *be the first time* B *attains a diameter of size* h.

(i) (Pitman [11]). $M(\delta_h)$ *and* $m(\delta_h)$ *are uniformly distributed on their respective ranges* $[0, h]$ *and* $[-h, 0]$.

(ii) (Imhof [7]). *The distribution of* $B(\delta_h)$ *is given by the* V*-shaped density function* $f_h(x) = \frac{|x|}{h^2}$, $|x| \le h$. *Consequently*, $E[\delta_h] = E[B^2(\delta_h)] = \frac{h^2}{2}$. *Similarly, for positive integer* h *and integer* $x \in [-h, h]$, *the probability that the simple random walk stopped at* δ_h *terminates at* x *is* $\frac{|x|}{h(h+1)}$.

Imhof (formula (2.1), [7]) identifies the joint distribution of $(\delta_h, B(\delta_h))$ and obtains [formula (2.2)] the V-shaped marginal density of $B(\delta_h)$. Here is a direct argument for random walk, from which the statement for B follows by a standard limiting argument: for $x \in \{1, 2, \dots, h\}$ (similarly for $x \in \{-h, -h+1, \dots, -1\}$), termination occurs at x if and only if $x - h$ is reached before x and then x is reached before going below $x - h$. Since the probability of the second stage is independent of x, the probability of terminating at x is proportional to the probability of the first stage, which by Lemma 1(i) is $\frac{x}{x+(h-x)} = \frac{x}{h}$. The consequence for $E[\delta_h]$ is implied by $B^2(t) - t$ being a mean-zero martingale, and the V-shaped density having variance $\frac{h^2}{2}$.

Pitman ([11], page 322) infers Lemma 2(i) from (ii) in the framework of Brownian motion on a circle, when first covering the entire circle. Here is a direct argument: For $0 < x < h$, $M(\delta_h) \le x$ if and only if B reaches $x - h$ before x. By Lemma 1(i), this event has probability $\frac{x}{h}$.

COROLLARY 3. (i) *The expected additional time Brownian motion needs to increase its diameter from* h_1 *to* $h_2 > h_1$ *is* $E[\delta_{h_2}] - E[\delta_{h_1}] = \frac{h_2^2 - h_1^2}{2}$.

(ii) $E[\mathcal{T}_d] = E[\delta_{2d}] + E[\tau_d] = 3d^2$.

(iii) $E[D(\mathcal{T}_d)] = 3d$.

PROOF. Claim (i) follows directly from Lemma 2(ii). By the two-stage description of $\mathcal{T}_d$ which follows its definition (1.2), $\mathcal{T}_d$ is the sum of δ_{2d} and a random time distributed like τ_d. Claim (ii) now follows from Lemmas 1(ii) and 2(ii) by taking expectations. The diameter at time $\mathcal{T}_d$ consists of the initial $2d$ plus the increment obtained during the second stage. By Lemma 1(ii) this increment has mean d, verifying claim (iii). □

The next lemma is instrumental in proving (in the following Proposition 1) that the process $Q_{c,1/(2c)}$ is a supermartingale.

LEMMA 3. (i) (Paul Lévy [9]). *The processes* $\{|B(t)| : t \ge 0\}, \{M(t) - B(t) : t \ge 0\}, \{B(t) - m(t) : t \ge 0\}$ *are identically distributed.*

(ii) *The processes* $\{B^2(t) - t : t \ge 0\}, \{(M(t) - B(t))^2 - t : t \ge 0\}, \{(B(t) - m(t))^2 - t : t \ge 0\}$, *adapted to the filtration of* B, *are mean-zero martingales.*

(iii) *The processes* $\{\max(B(t), 0)^2 - t : t \ge 0\}, \{\min(B(t), 0)^2 - t : t \ge 0\}$ *are supermartingales.*

PROOF. Assuming that the martingale nature of $\{B(t)^2 - t\}$ is well known, statement (ii) follows from (i). To prove Lemma 3(iii), let $0 \le t < s$. In terms of the stopping time $\rho = \min(s, \inf\{u : u > t, B(u) \ge 0\})$, there are three possible cases to consider: $\{B(t) \ge 0\}$, $\{B(t) < 0, \rho = s\}$ and $\{B(t) < 0, t < \rho < s\}$. In the first case, by statement (ii) of the lemma, $E[\max(B(s), 0)^2 - s|\mathcal{B}_t] \le E[B^2(s) - s|\mathcal{B}_t] = B^2(t) - t = \max(B(t), 0)^2 - t$ a.s. For the other two cases, condition on $\mathcal{B}_\rho$ (which contains $\mathcal{B}_t$) to obtain

$$\begin{aligned}
&E[\max(B(s), 0)^2 - s|\mathcal{B}_t] \\
&\quad = E\big[E[\max(B(s), 0)^2 - (s - \rho)|\mathcal{B}_\rho]|\mathcal{B}_t\big] - t - E[\rho - t|\mathcal{B}_t] \\
&\quad < E\big[E[\max(B(s), 0)^2 - (s - \rho)|\mathcal{B}_\rho]|\mathcal{B}_t\big] - t.
\end{aligned} \tag{2.2}$$

In the second case, $\max(B(s), 0)^2 - (s - \rho) = 0 = \max(B(t), 0)^2$ a.s. In the third case, by statement (ii) of the lemma, $E[\max(B(s), 0)^2 - (s - \rho)|\mathcal{B}_\rho] \le E[B^2(s) - (s - \rho)|\mathcal{B}_\rho] = B^2(\rho) = 0 = \max(B(t), 0)^2$ a.s. So in each of these two cases we obtain that the RHS of (2.2) is bounded from above by $\max(B(t), 0)^2 - t$. $\square$

PROPOSITION 1. *For $d = \frac{1}{2c}$, the process $Q = Q_{c,d} = Q_{c,1/(2c)}$ has the following properties*:

(i) *$\forall t \ge 0$, $Q(t) \ge \Pi(t)$ a.s.*

(ii) *$Q(0) = E[\Pi(\mathcal{T}_{1/(2c)})] = E[Q(\mathcal{T}_{1/(2c)})]$. Moreover, $\Pi(\mathcal{T}_{1/(2c)}) = Q(\mathcal{T}_{1/(2c)})$ a.s.*

(iii) *Q is a supermartingale and $\{\check{Q}(t) = Q(t \wedge \mathcal{T}_{1/(2c)}) : t \ge 0\}$ is a martingale* (*w.r.t. the filtration $\{\mathcal{B}_t\}$ of the underlying Brownian motion*).

PROOF. Substituting $\frac{1}{2c}$ for d in (2.1), a straightforward calculation yields

$$\begin{aligned}
&Q(t) - \Pi(t) \\
&\quad = \begin{cases}
0, & G(t) \ge \dfrac{1}{2c}, \\[2mm]
c\Big[\dfrac{1}{4}\Big(\dfrac{1}{c} - D(t)\Big)\Big(\dfrac{3}{c} - D(t)\Big) & \\
\qquad\qquad + \Big(\dfrac{D(t)}{2} - G(t)\Big)^2\Big], & D(t) < \dfrac{1}{c}, \\[2mm]
c\Big[\dfrac{1}{2c} - G(t)\Big]^2, & D(t) \ge \dfrac{1}{c}, G(t) < \dfrac{1}{2c},
\end{cases}
\end{aligned} \tag{2.3}$$

which is nonnegative, thus verifying claim (i). Since $G(0) = D(0) = \Pi(0) = 0$, it follows from (2.3) that $Q(0) = \frac{3}{4c}$. The first equality in claim (ii) now follows by applying (ii) and (iii) of Corollary 3 (with $d = \frac{1}{2c}$). To prove the third, a fortiori the second equality in claim (ii), just note that by definition $G(\mathcal{T}_{1/(2c)}) = \frac{1}{2c}$ and $D(\mathcal{T}_{1/(2c)}) \geq \frac{1}{c}$.

To establish claim (iii), the time axis $[0, \infty)$ will be partitioned into a sequence of intervals with suitably chosen stopping times as their end-points. The process $Q(\cdot)$ will then be represented in an appropriate form, tailored for the application of Lemma 3, over each of these subintervals. To exhibit this partition, fix an arbitrary $f \in (0, d)$ and inductively define an increasing sequence of stopping times as follows: $\tau_0 = 0$ a.s., τ_1 is the first time B achieves diameter $2d$ and $\tau_2 = \mathcal{T}_d$. Note that $B(\tau_2)$ is an end-point of the a.s. nonempty *central interval* $(m(\tau_2) + d, M(\tau_2) - d)$. Let τ_3 be the earlier between the next time B reaches the other end-point of the central interval or the gap decreases to f. Generically now, for $n \geq 3$, if $B(\tau_{n-1})$ and $B(\tau_n)$ are the end-points of the central interval $(m(\tau_n) + d, M(\tau_n) - d)$, define τ_{n+1} similarly to τ_3. If, on the other hand, the gap at $B(\tau_n)$ is f, let τ_{n+1} be the first time after τ_n at which the gap reaches d again. We now use (2.3) to represent $Q(\cdot)$ over each of the partition intervals $[\tau_{n-1}, \tau_n)$ in a form conducive to the application of Lemma 3.

Between times τ_0 and τ_1, Q is equal to the martingale [see Lemma 3(ii)] $\frac{3}{4c} + \frac{c}{2}\{[(M(t) - B(t))^2 - t] + [(B(t) - m(t))^2 - t]\}$. Resorting to the short-hand $t^* = \max(\tau_1, \min(\tau_2, t))$, between times τ_1 and τ_2, Q is equal to the martingale defined as Q up to τ_1 and, thereafter [see again Lemma 3(ii)], $\frac{1}{4c} + B(t^*) - m(\tau_1) + c[(M(t^*) - B(t^*))^2 - t^*]$ or its mirror image, depending on $B(\tau_1)$ being $M(\tau_1)$ or $m(\tau_1)$. A similar representation of $Q(\cdot)$ is readily available for the time increments during which the gap increases from f to d. Finally, consider a time τ_n at which B is at an end-point of the then-central interval. Letting $t^* = \max(\tau_n, \min(\tau_{n+1}, t))$, from time τ_n to time τ_{n+1}, the process Q is equal to the supermartingale defined as Q up to τ_n, and thereafter [see Lemma 3(iii)], $M(\tau_n) - m(\tau_n) - c\tau_n + c[\max(W(t^*), 0)^2 - (t^* - \tau_n)]$, where, depending on $B(\tau_n)$ being $M(\tau_n) + d$ or $m(\tau_n) - d$, $W(\cdot) = B(\cdot) - B(\tau_n)$ or its mirror image.

Since $\bigcup_{n=1}^{\infty} [\tau_{n-1}, \tau_n) = [0, \infty)$ a.s., $Q(\cdot)$ is a supermartingale throughout. $\square$

Concluding the proof of Theorem 1. As argued prior to the statement of Lemma 1, Proposition 1 establishes the optimality of $\mathcal{T}_{\frac{1}{2c}}$ for the current c-problem.

By Corollary 3, $E[\Pi(\mathcal{T}_d)] = E[D(\mathcal{T}_d)] - cE[\mathcal{T}_d] = (2d + d) - c(\frac{(2d)^2}{2} + d^2) = 3d - 3cd^2$, which is uniquely maximized at $d = \frac{1}{2c}$. Thus $\mathcal{T}_{1/(2c)}$ is the unique optimal stopping time within the family $\{\mathcal{T}_d : d > 0\}$.

Claims (ii) and (iii) of Theorem 1 follow straightforwardly from Lemma 2.

PROOF OF COROLLARY 1. As argued in the last paragraph of the Introduction, it is enough to consider martingales of the form $\{B(t) : t \leq T\}$, where T is a stopping time with $E[T] = E[(B(T))^2] < \infty$. Let $\sigma = \sqrt{E[T]}$ and consider the

c-problem with $c = \frac{\sqrt{3}}{2\sigma}$. Then

$$\begin{aligned} E[D(T)] &= \big(E[D(T)] - cE[T]\big) + cE[T] \\ &\le E\big[D(\mathcal{T}_{1/(2c)})\big] - cE\big[\mathcal{T}_{1/(2c)}\big] + c\sigma^2 \\ &= \frac{3}{4c} + c\sigma^2 = \frac{3}{4\sqrt{3}/(2\sigma)} + \frac{\sqrt{3}}{2}\sigma = \sqrt{3}\sigma. \end{aligned} \tag{2.4}$$

That the $\sqrt{3}$-bound is attained by $\{B(t) : t \le \mathcal{T}_d\}$ for any $d > 0$ follows from Corollary 3. □

3. An open problem: the spider process. Larry Shepp has recently reminded us that the $\sqrt{3}$-inequality treated here is a special case of the so-called *spider* problem raised some time ago by the first author. Informally speaking, Brownian motion may be viewed as an absolute value of Brownian motion, each of whose excursions is assigned a random sign. The spider process (sometimes called the *Walsh* process) with $n \ge 3$ rays emanating from the origin is the extension from Brownian motion ($n = 2$) to an n-valued sign. (Thus, $n = 4$ corresponds to randomly switching at visits to zero between an absolute value of BM on the y axis and another on the x axis.) The maximal distance from the origin in the spider process is simply the maximal absolute value of Brownian motion, independently of n. On the other hand, the sum of the distances from the origin along the rays reduces in the case $n = 2$ to the diameter of Brownian motion studied here. The maximization of the expected value of this sum of distances when $n \ge 3$ seems harder to handle and evidently requires new ideas.

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