Goodness-of-fit test for density estimation with directional data

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a joint work with *G. Boente* and *W. Gonzalez-Manteiga.*

Objective

The main idea of this work is develop a hypothesis test when the variables belong on the sphere. We will assume that we have a random sample $\mathbf{x}_1, \ldots, \mathbf{x}_n$ taking values in the $d-$dimensional unit sphere S^d in $\mathbb{R}^{d+1}$ with probability density function $f(x)$.

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We are interested in testing the following hypothesis

$H_o : f(x) \in \mathcal{G}$ for some $\beta_o \in \mathcal{B}$ against $H_1 : f(x) \notin \mathcal{G}$

Test

To test this hypothesis

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$$f_n(x) = \frac{c(h)}{n} \sum_{i=1}^{n} K\left(\frac{1 - x'\mathbf{x}_i}{h^2}\right)$$

where $c(h)$ is a normalizing constant given by $\frac{1}{c(h)} = \int_{S^d} K((1 - x'y)/h^2)\omega_d(dx)$.

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Then, we compare the estimators trough

- The L^2 distance: $\int (f_n(x) - f_{\widehat{\beta}}(x))^2 \, \omega_d(dx)$
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We assume that

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$$\widehat{\beta} - \beta_o = O_p(n^{-1/2}).$$

Some standard conditions on the kernel and the bandwidth.

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$$g_d = \begin{cases} \int_0^\infty \rho^{-1/2} K(\rho) [K(r+\rho-2(r\rho)^{1/2}) + K(r+\rho+2(r\rho)^{1/2})] d\rho & d=1 \\ \int_0^\infty \rho^{d/2-1} K(\rho) \int_{-1}^1 (1-\theta^2)^{(d-3)/2} K(r+\rho-2\theta(r\rho)^{1/2}) d\theta d\rho & d>1 \end{cases}$$

Asymptotic behavior under a sequence of local alternatives for the L^2 distance

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If we consider the sequence of regular alternatives represented by

$$H_{1c} : f(x) = f_{\beta_o}(x) + \frac{1}{\sqrt{nh^{d/2}}} \Delta(x)$$

where $\int \Delta(x) \omega_d(dx) = 0$.

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where $\int \Delta(x)\omega_d(dx) = 0$.
Under some conditions we have that

$$nh^{d/2} \left(T_n - \frac{b}{nh^d} \right) \xrightarrow{\mathcal{D}} N\left(\int \Delta^2(x)\omega_d(dx), 2\sigma^2 \right) \text{ under } H_{1c}$$

where b and σ^2 are defined in the previous result.

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If $h \to 0$, $\sqrt{n}h \to \infty$ and K satisfy standard conditions, Mason (2000) proved, in the case of real variables, that

$$\sqrt{n}\left(\int |f_n(x) - f(x)| dx - E\int |f_n(x) - f(x)| dx\right) \xrightarrow{\mathcal{D}} N(0, \sigma^2(K))$$

where $\sigma^2(K)$ is a constant depending of the kernel. (Giné, Mason and Zaitsev (2003) improved this result.)

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The important property in this results is the rate of convergence.

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Now, we are working in this problem

$$\sqrt{n}\,(W_n - E(W_n)) \xrightarrow{\mathcal{D}} N\left(0, \sigma^2\right) \text{ under } H_0 \text{ ???}$$

Parametric Bootstrap: composite hypothesis

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- The asymptotic results suggest the use of bootstrap methods to calibrate the test when considering the L^1 distance.
- Moreover, for the L^2 test statistic, the rate of convergence is slow and so, we may expect that the normal approximation will not work well for moderate sample sizes.
- In order to provide an alternative to the asymptotic distribution of test statistics, we will study a bootstrap procedure.

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Step 2 Compute T_n and W_n with the bootstrap sample and call it T_n^* and W_n^*, i.e.

$$T_n^* = \int \left(f_n^*(x) - K_h f_{\widehat{\beta}^*}(x)\right)^2 \omega_d(dx)$$

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Let $t_{n,\alpha}$ (or $w_{n,\alpha}$) the upper $\alpha-$percentil of the empirical bootstrap distribution, then we will reject H_o if $T_n > t_{n,\alpha}$. or ($W_n > w_{n,\alpha}$.)

Validity of bootstrap procedure

The asymptotic distribution of T_n^* under the null hypothesis

Under some conditions, we have that the bootstrap distribution of T_n, converges to the asymptotic null distribution of T_n.

$$nh^{d/2}\left(T_n^* - \frac{b}{nh^d}\right) \xrightarrow{\mathcal{D}} N(0, 2\sigma^2) \text{ under } H_o$$

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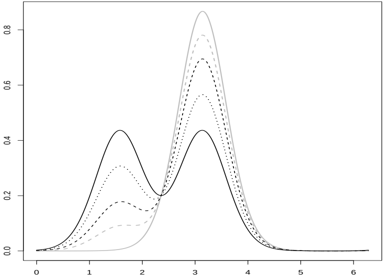
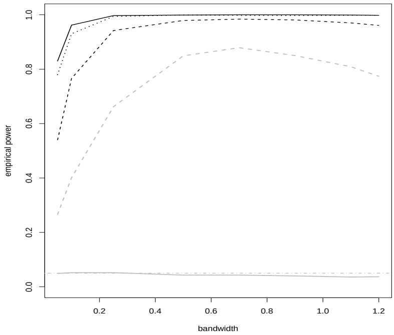
To study the performance of test when the null hypothesis is false, we select a set of alternatives from $(1-\delta)f_{\pi,5} + \delta f_{\pi/2,5}$ with $\delta = 0.1, 0.2, 0.35, 0.5$ where $f_{\pi,5}$ is the density under H_o and $f_{\pi/2,5}$ is a von Mises variable with mean $\mu = \pi/2$ and $\kappa = 5$.

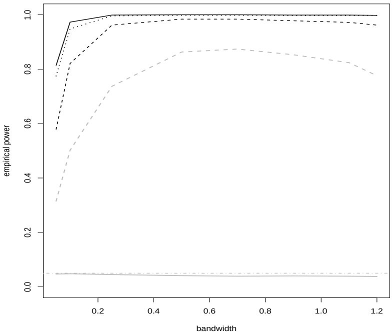


Simulations Results:

Test L^2

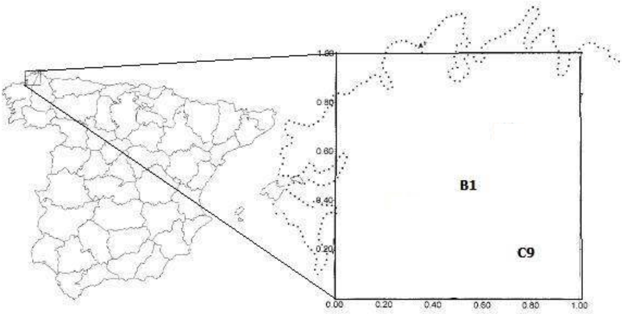


Test L^1



Example

This dataset consists on the directions by winds in two meteorologic stations of Galicia, Spain in August 2009.



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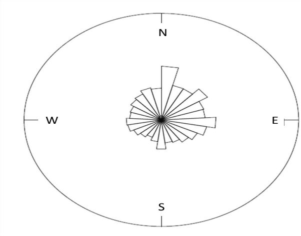
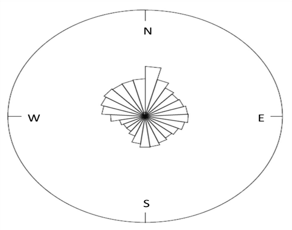


a) Station B1



b) Station C9

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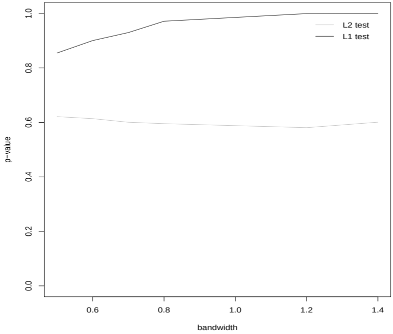
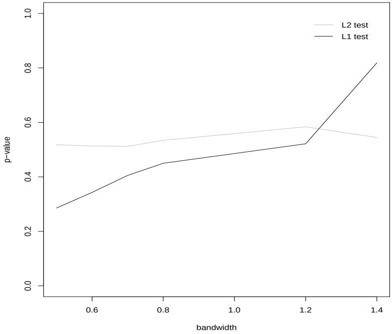
	estimators	
stations	$\widehat{\mu}$	$\widehat{\kappa}$
B1	0.6086	0.645
C9	0.181	0.4697

Example: stations B1 and C9





a) p-values of Station B1 *b*) p-values of Station C9

That's all. Thank you!